Phased Synthesis of Divide and Conquer Programs

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Abstract

We propose a fully automated method that takes as input an iterative or recursive reference implementation and produces divide-and-conquer implementations that are functionally equivalent to the input. Three interdependent components have to be synthesized: a function that divides the original problem instance, a function that solves each sub-instance, and a function that combines the results of sub-computations. We propose a methodology that splits the synthesis problem into three successive phases, each with a substantially reduced state space compared to the original monolithic task, and therefore substantially more tractable. Our methodology is implemented as an addition to the existing synthesis tool Parsynt, and we demonstrate the efficacy of it by synthesizing highly nontrivial divide-and-conquer implementations of a set of benchmarks fully automatically.

CCS Concepts: • **Theory of computation** → **Program reasoning**; **Divide and conquer**; Parallel computing models; • **Software and its engineering** → **Automatic programming**.

Keywords: Program Synthesis, Divide and Conquer

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1 Introduction

A divide-and-conquer computation decomposes a problem instance into several smaller sub-problems, solves each *independently*, and then combines the results to solve the original problem. It may produce better solutions for algorithmic problems by (i) improving the asymptotic complexity of the computation, for all program inputs or for generic subsets of them, or (ii) creating the potential for leveraging parallelism, since independent subtasks can be easily parallelized with good speedups. Writing a good divide-and-conquer algorithm is often non-trivial and can sometimes be quite tricky. Every undergraduate algorithms textbook has a chapter on divide-and-conquer and attempts to teach computer science students how to do the task by example. There are sometimes several instances of a divide-and-conquer solution for a given problem (an example follows in Section 2), and the specific usage determines the preferred solution from the pool of candidates. Automated synthesis can therefore offer a lot of utility in this problem space.

This paper proposes a systematic and automatable way of inferring a divide-and-conquer algorithm from an input reference implementation. The target divide-and-conquer algorithms adhere to the diagram in Figure 1. The input is an existing (iterative or recursive) implementation of a function $f : \mathcal{S} \mapsto D$, which is a single pass function over a collection (of general type $\mathcal{S}$). The output is a divide-and-conquer implementation of the same function. More specifically, a triple of functional components $(\triangledown, \mathbf{f}, \odot)$ is synthesized where $\triangledown : \mathcal{S} \to \mathcal{S}^n$ is an n-way *divide operator* ($n = 2$ in the figure), $\odot : D^n \to D$ is the complementary *join operator*, and $\mathbf{f}$ computes f and potentially some extra information which is strictly required for $\odot$ to recover $f(l)$ from the partial computations of subtasks. $\mathbf{f}$ is a *lifting* (in the standard category theory sense) of f.

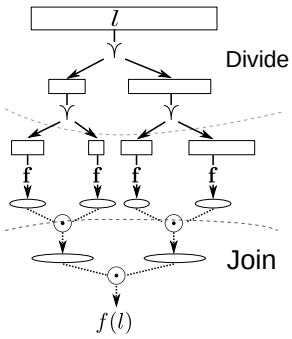

Figure 1. D&C Schema

The main restrictions of this family, compared to broadly understood divide-and-conquer algorithms, are: (i) the divide function $\triangledown$ has to be recursively applicable to an input collection, dividing it into smaller and smaller pieces, for an arbitrary number of calls, and (ii) the computation performed in each subproblem is a *lifting* of f. Yet, the model is very general and admits many interesting divide-and-conquer algorithms from the literature. In particular, it subsumes the `dc1` category from [24] which proposes a manual methodology for producing such algorithms, and it is substantially

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more expressive than any class of divide-and-conquer considered for automated synthesis so far. MapReduce [7], for example, is a limited class of divide-and-conquer algorithms that has been targeted by automated synthesis successfully before [8, 9, 11, 22, 23]. The more general class we undertake is strictly more challenging to synthesize since three interdependent unknown code components $(\curlyvee, \mathbf{f}, \odot)$ need to be synthesized simultaneously. In [11, 22, 23], the target of synthesis is only the join operator $\odot$ (i.e. the *reduction* in MapReduce terminology). In [8, 9], the pair $(\mathbf{f}, \odot)$ was targeted through the simplifying assumption that $\curlyvee$ defaults to a simple sequence split operator, which is the inverse of sequence concatenation, i.e $\curlyvee(x \bullet y) = (x, y)$; this is the default assumption for all MapReduce frameworks.

We propose a *phased* synthesis procedure that breaks up the task of synthesizing $(\curlyvee, \mathbf{f}, \odot)$ into three separate synthesis subtasks. Figure 2 illustrates the workflow of our methodology. The first observation is that instead of synthesizing $\curlyvee$, one can synthesize a *divide predicate*, namely a specification for the acceptable outcome of $\curlyvee$. A predicate p is a valid divide predicate if and only if the results of the computations performed on the parts that satisfy p can be combined correctly. In other words, *there exists* a function $\odot$ such that, if a an input z is split into two parts x and y such that $p(x, y)$ holds, then $f(z) = f(x) \odot f(y)$. Our approach exploits the fact that $\odot$ has to only exist and need not be determined while p is being synthesized.

If p is successfully synthesized, then $\curlyvee$ is synthesized such that its output adheres to what p specifies. If the synthesis of p fails, then the reason for this failure may be that extra information, which is currently missing from what f computes, is required for the existence of $\odot$. Therefore, an attempt is made to *lift* f, and it is replaced by a new function $\mathbf{f}$, which additionally computes the missing information. Then, the attempt to synthesize p is repeated for the new function $\mathbf{f}$.

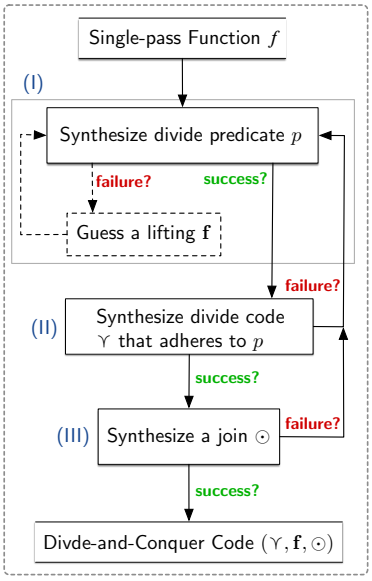


Figure 2. Phased Synthesis Schema

The synthesis of $\curlyvee$ may also fail. In general, it is not always possible or feasible to synthesize a piece of code (i.e. $\curlyvee$) that adheres to a given specification (i.e. p). In this case, a new p is requested with the hopes that a change in the predicate will lead to a successful step (II). Once $\curlyvee$ is synthesized, the algorithm proceeds to synthesize $\odot$ as the only remaining unknown, which is guaranteed to exist at this point. If this is successful, the procedure concludes and $(\curlyvee, \mathbf{f}, \odot)$ is returned.

The synthesis loop is further constrained to only explore implementations that are at least as efficient as the input reference implementation, so that no useless divide-and-conquer solutions are generated. One can iterate through the loop to find a first valid solution, and continue to enumerate several valid solutions.

Once the synthesis problem is decomposed as depicted in Figure 2, the synthesis of $\curlyvee$ and $\odot$ (boxes (II) and (III)) can be performed in a relatively standard way through syntax-guided synthesis, since each phase performs the synthesis of a single unknown code component (Section 7). We propose a novel algorithm for the synthesis of the *divide predicate* p (box (I)), which also predicts if a lifting of f will be required and produces the lifting. This algorithm, in the spirit of *deductive synthesis* [17], simultaneously infers p and any required lifting $\mathbf{f}$ that would guarantee the existence of a join implementation $\odot$, without the need to implement the dashed loop (for guessing $\mathbf{f}$) depicted in Figure 2.

In Figure 2, the input is a single-pass function f over a collection. In our technique, we accept an iterative or recursive implementation as input and produce an equivalent single-pass function f automatically. This step is described in [10]. In summary, in this paper:

- We lay out the theoretical foundations to reduce the problem of divide-and-conquer synthesis from the specification of Figure 1 to one more amenable to automation (Section 4).
- We propose a phased synthesis algorithm that synthesizes the triple of unknowns $(\curlyvee, \mathbf{f}, \odot)$ in three different stages employing both syntax-guided synthesis and deductive synthesis techniques (Section 5).
- We propose a novel algorithm based on deductive synthesis that can efficiently discover two unknowns, a divide predicate and a lifting of f (Section 6). Our proposed automated lifting algorithm surpasses previously known algorithms [8, 9] in that it can infer conditional accumulators to extend the signature of the function which were not possible before.
- We illustrate through a set of benchmarks that an implementation of our proposed approach can synthesize highly nontrivial divide-and-conquer solutions based on simple input implementations.

2 Motivating Example

We use an example to illustrate the types of programs that our approach can synthesize automatically. Additionally, the example underlines the following two observations: (i) there are often several acceptable divide-and-conquer implementations of a given function, and (ii) synthesizing a divide-and-conquer solution is not solely about discovering

the division and join operators, but may also require a lifting of the original input code.

Consider a set of (input) points on a 2D plane. A point is *Pareto optimal* if all the other points are either below or to the left of this point. Let $p.x$ and $p.y$ denote the coordinates of point p. Formally:

$$p \succcurlyeq p' \iff p.x \geq p'.x \vee p.y \geq p'.y$$
$$\text{POP}(X) = \{p \in X \mid \forall p' \in X, p \succcurlyeq p'\}.$$

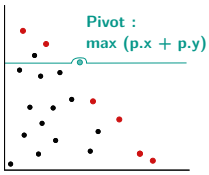
The code in Figure 3 computes POP(X) where the input X is a list of points. At each iteration, the set of optimal points is updated by maintaining the points that remain optimal with respect to the new input, and adding the new input if it is optimal with respect to all currently optimal points. If the divide operator is taken as the trivial split of the input list, then the correct join matching this divide has a quadratic time complexity. By the Master Theorem [5], the complexity of the resulting *naive* divide-and-conquer algorithm is $O(n^2)$, which matches that of the original input implementation.

```
List<Point> l = [];
List<Point> tmp = [];
for(i = 0; i < n; i++) {
  Point p = A[i];
  bool b = true;
  tmp = [];
  for(e in l) {
    if(e ≻ p)tmp.append(e);
    b = b && (p ≻ e); }
  if (b) tmp.append(p);
  l=tmp;
}
```

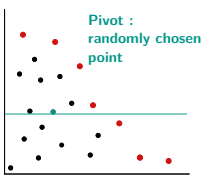
Figure 3. Single pass POP

Other *divide functions* yield algorithms with lower asymptotic complexities. We briefly introduce three such algorithms with a two-way divide, a two-way divide with *lifting* and a three-way divide, all of which our tool Parsynt can automatically synthesize.

The solution with a two-way divide is illustrated on the right: a (pivot) point p is chosen by taking the point with the maximum sum of coordinates, which is guaranteed to be Pareto optimal. The point set is then partitioned into two sets: the set of points (vertically) above and strictly below p. The optimal points of the original point set is then the concatenation of the lists of optimal points from each partition.



If the pivot is chosen at random, and therefore not guaranteed to be Pareto optimal, then the Pareto optimal points of the top partition all remain optimal. But of the ones in the bottom partition, those which are to the left of the rightmost point of the top partition have to be removed. This cannot be done without a *lifting*. Some additional information, for example the rightmost point of each partition, is needed so that the join can correctly combine the two Pareto optimal sets by pruning the result from the bottom partition.



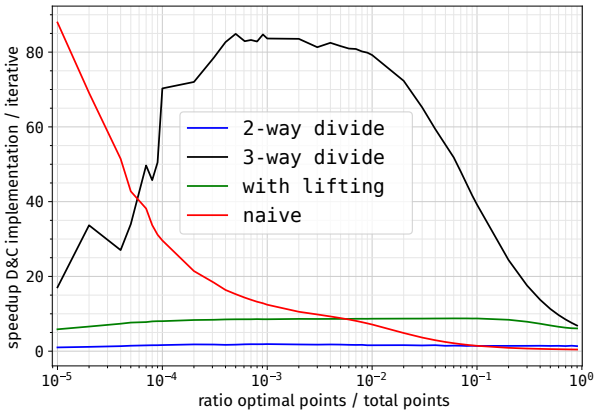
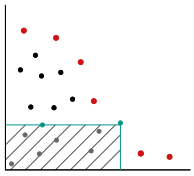


Figure 4. Speedups of sequential divide-and-conquer implementations of POP relative to implementation in Figure 3

Finally, in the implementation with a three-way divide, a point is chosen at random and the rightmost point above this point is chosen as the pivot. The point set is partitioned into three subspaces (as illustrated on the right): the points above, to the right, and below and to the left of the pivot. The third partition (hatched) does not contain any Pareto optimal point. The Pareto optimal points for the other two partitions are optimal for the original point set and therefore the join operator can simply concatenate the results from these two partitions.



All algorithms, except the naive one, partition the space in linear time, and join the results in constant or linear time, which puts them in the same asymptotic complexity class $O(n \log n)$. However, the performance of these solutions varies significantly depending on the composition of the input data; specifically, the ratio of the Pareto optimal points to the total number of points. The graph in Figure 4 illustrates the speedups of the different divide-and-conquer implementations relative to the input implementation of Figure 3. The horizontal axis is the ratio of optimal points in the input list (of size 2×10^5). When the ratio of optimal points is very small, the naive implementation performs significantly better than all other implementations, with speedups reaching 80x. When there are very few Pareto optimal points, the (quadratic) join operator defaults to a constant time complexity. As the ratio of optimal points increases, the performance of the naive implementation decreases to drop below all the other algorithms. Our tool produces all algorithms automatically and the user selects the best for their specific usage.

3 Background and Notation

Let *Sc* be a type that stands for any scalar type used in typical programming languages, such as `int` and `bool`, whenever the specific type is not important in the context. Scalars are

assumed to be of *constant* size, and conversely, any constant-size representable data type is assumed to be scalar. Consequently, all operations on scalars are assumed to have constant time complexity. Type $\mathcal{S}$ defines the set of all *sequences* of elements of type Sc. The concatenation operator $\bullet : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ defined over sequences is associative. The sequence type stands in for *arrays*, *lists*, or any collection data type that admits a linear iterator and an *associative* composition operator. A function $h : \mathcal{S} \to D$ is rightwards iff there exists a binary operator $\oplus : D \times Sc \to D$ such that for all $x \in \mathcal{S}$ and $a \in Sc$, we have $h(x \bullet [a]) = h(x) \oplus a$. A rightwards function can be defined by a left fold over a sequence: $h(x) = \textbf{foldl}\ \oplus\ h([])\ x$. A leftward function is defined analogously using the recursive equation $h([a] \bullet x) = a \otimes h(x)$. A function is *single-pass* if it is leftwards or rightwards.

4 Decomposing D&C Specification

The input to our approach is an implementation of a single-pass function $f : \mathcal{S} \to D$. To accommodate generic reference implementations, which may be a (nested) loop or a recursive function performing multiple passes over the input data, we propose a (source-to-source) translation in [10]. This translation converts an arbitrary iterative or recursive input implementation to a single pass recursive function f.

The goal of this section is to start with a generic specification for divide-and-conquer and transform it to a *tractable* specification for search-based synthesis. A general divide-and-conquer algorithm comprises of a *divide* function $\curlyvee : \mathcal{S} \to \mathcal{S}^c$ and a *join* function $\odot : D^c \to D$ (with $c > 1$) that satisfy the specfication:

$$\Psi(\odot, \curlyvee) \equiv \forall z \in \mathcal{S} : \\ f(z) = \odot\big(f(\curlyvee(z).1), f(\curlyvee(z).2), \ldots, f(\curlyvee(z).c)\big) \quad (1)$$

Since $\odot$ and $\curlyvee$ must be computable, the solution space for them is the set of recursive functions. To accommodate full automation, we focus on a slightly more limited universe of divide functions, namely those that *partition* the input space.

To simplify the formal notation, we restrict c to be precisely 2. All formal statements stated and proved in this paper generalize to any value for c, and therefore this does not cause a loss in generality. The stronger specification for *partition* divide-and-conquer algorithms is:

$$\Psi^\circ(\odot, \curlyvee) \equiv \forall z \in \mathcal{S} :\ f(z) = f(\curlyvee(z).1) \odot f(\curlyvee(z).2) \\ \wedge\ \widetilde{z} = \widetilde{\curlyvee(z).1} \cup \widetilde{\curlyvee(z).2} \quad (2)$$

where $\widetilde{z}$ denotes the set of elements of sequence z.

Note that both Ψ and Ψ° admit trivial (useless) solutions. For example, a valid solution for $\curlyvee$ is to divide a sequence s into the sequence s and the empty sequence $[]$, and let $\odot$ return its first component. To rule these out, we add a constraint on the sizes of individual outputs generated by $\curlyvee$ over a universe of inputs. It requires the existence of at least one input which would divide a list of length $m + k$ into two sublists of arbitrary sizes m and k. Formally:

$$\chi(\curlyvee) \equiv \forall m, k \in \mathbb{N}, \exists z \in \mathcal{S} : |\curlyvee(z).1| = m \wedge |\curlyvee(z).2| = k$$

Note that this can be extended to multi-way divides in a straightforward way. Combining Ψ° from Equation 2 with χ will result in our first concrete specification with non-trivial solutions:

$$\Psi^\bullet(\curlyvee, \odot) \equiv \Psi^\circ(\curlyvee, \odot) \wedge \chi(\curlyvee) \quad (3)$$

$\Psi^\bullet$ (a strict strengthening of Ψ) is the precise specification we aim to use for divide-and-conquer synthesis. Yet this specification defines a huge (intractable) search space for existing search-based program synthesis techniques. We propose a way to decompose this specification such that $\curlyvee$ and $\odot$ can be synthesized *independently*, even though they are related through $\Psi^\bullet$.

Key insight. The most straightforward division operation is the inverse of sequence concatenation, that is, the sequence z is divided into any pair of sequences z_1 and z_2 such that $z = z_1 \bullet z_2$. The key observation is that a general divide $\curlyvee$ satisfying $\Psi^\bullet$ can be defined as a composition of a sequence permutation function and this trivial divide. That is, if $\Psi^\bullet(\curlyvee, \odot)$ then there exists a permutation function $\pi : \mathcal{S} \to \mathcal{S}$ such that $\forall z \in \mathcal{S} : z = \pi(\curlyvee(z).1 \bullet \curlyvee(z).2)$. This a simple consequence of the constraint $\widetilde{z} = \widetilde{\curlyvee(z).1} \cup \widetilde{\curlyvee(z).2}$.

The insight leads to the specification below, which makes use of a predicate $p : \mathcal{S} \times \mathcal{S} \to \text{Bool}$ and a permutation function π instead of the divide function $\curlyvee$:

$$\Phi(\pi, p, \odot) \equiv \chi^\bullet(p) \wedge \forall (x, y) \in \mathcal{S}^2 : \\ p(x, y) \Rightarrow f(\pi(x \bullet y)) = f(x) \odot f(y) \quad (4)$$

where χ is reformulated for p as

$$\chi^\bullet(p) = \forall m, k \in \mathbb{N}, \exists x, y \in \mathcal{S}^2 : |x| = m \wedge |y| = k \wedge p(x, y)$$

The new specification is only a reframing of our problem, and as the following theorem states, $\Psi^\bullet$ can be used instead of Φ without a compromise.

Theorem 4.1. *The specifications $\Psi^\bullet$ (defined in Equation 3) and Φ (defined in Equation 4) are mutually realizable.*

So far, we have reformulated the problem of synthesizing a divide and a join operation to a different yet *equirealizable* problem of synthesizing a predicate p and a join operation. This is an intermediate step that facilitates the independent algorithmic synthesis of p and $\odot$, in contrast to the monolithic task that is put forward by the specification Φ. First, we discuss how this can be achieved through a specialization of the synthesis problem.

4.1 Permutation Invariance

If f is not sensitive to the order of elements in its input sequence, then π can be eliminated from Φ.

Definition 4.2 (Permutation invariant)**.** A function f is *permutation invariant* iff for all permutation functions π and all lists $x \in \mathcal{S}$, $f(x) = f(\pi(x))$.

If f is permutation invariant and $(\pi, p, \odot)$ is a solution for Φ, then $(\pi', p, \odot)$ for any permutation function π' is also a valid solution to Φ. Therefore, we can simply replace π with the identity permutation and simplify Φ to:

$$\Phi^\bullet(p, \odot) \equiv \forall x, y \in \mathcal{S}^2 : \\ p(x, y) \Rightarrow f(x \bullet y) = f(x) \odot f(y) \wedge \chi^\bullet(p) \quad (5)$$

Theorem 4.3. *If the function f is permutation invariant then the specifications $\Phi^\bullet$ and $\Psi^\bullet$ are mutually realizable.*

An insight from the proof (in Appendix B.2) is the necessary relation between p and $\curlyvee$, which must satisfy $\forall z \in \mathcal{S}, p(\curlyvee(z).1, \curlyvee(z).2)$.

The practicality of $\Phi^\bullet$ (over $\Psi^\bullet$) is that without π, one can synthesize p and $\odot$ in two independent (synthesis) steps as discussed in Sections 5 and 6.

4.2 Splitting Divides

The results in Section 4.1 are theoretically crisp, but seem restricted in the sense that they do not apply if the function is not permutation invariant. It turns out that solutions to specification $\Phi^\bullet$ go beyond divide functions for permutation invariant functions. Consider splitting divides as formally defined below.

Definition 4.4 (Splitting divide)**.** A divide function $\curlyvee : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is a splitting divide if $\forall z \in \mathcal{S} : z = \curlyvee(z).1 \bullet \curlyvee(z).2$.

A splitting divide $\curlyvee$ can also be synthesized (indirectly) through the specification $\Phi^\bullet$. The reason goes as follows. The restriction of $\Psi^\bullet$ to splitting divides is:

$$\Psi^\star(\curlyvee, \odot) \equiv \chi(\curlyvee) \wedge (z) = f(\curlyvee(z).1) \odot f(\curlyvee(z).2) \\ \wedge \ \forall z \in \mathcal{S} : \curlyvee(z).1 \bullet \curlyvee(z).2 = z \quad (6)$$

Proposition 4.5. *If $\Psi^\star$ is realizable then so is $\Phi^\bullet$.*

Proposition 4.5 concludes that whenever a splitting divide solution exists, $\Phi^\bullet$ also presents a corresponding solution. The converse does not hold; there may exist a solution for $\Phi^\bullet$ while no solution for $\Psi^\star$ exists. Note that splitting divides provide valid divide-and-conquer implementations for many instances where f is not permutation invariant.

So far we have argued how $\Phi^\bullet$ can be used to generate two different generic class of divide-and-conquer algorithms. Theoretically, there is no guarantee that either f is a permutation invariant or a splitting divide for f exists. Practically, however, we have not come across a single case for this. Nevertheless, to close the theoretical gap, in Appendix B.4, we discuss a general construction that translates an arbitrary function f to a permutation-invariant implementation of it $\check{f}$. We prove that realizability of $\Phi^\bullet$ for f implies realizability of $\Phi^\bullet$ for $\check{f}$. Therefore, if f is not permutation invariant and a splitting divide does not exist, then one can theoretically synthesize a divide-and-conquer solution for $\check{f}$ instead.

5 Synthesizing Divide-and-Conquer

The intention with the design of a divide-and-conquer implementation is to either obtain a better performing sequential algorithm or a parallelizable algorithm. In both cases, the resulting algorithm should not have a worse computational complexity than the input reference implementation. Note that the specification(s) from Section 4 carry no guarantees about the computational complexity of the synthesized code. We first introduce sufficient conditions that guarantee a reasonable computational complexity for the synthesized program. Then, we provide a refinement of the schema in Figure 2 that incorporates these complexity constraints and makes explicit use of specifications introduced in Section 4.

5.1 Complexity of Divide-and-Conquer

The time complexity of synthesized divide-and-conquer algorithms can be measured through the Master Theorem [5]. Under the assumption that the divide operator is *balanced*, the time complexity is defined through the recurrence $T(n) = kT(n/c) + w(n)$, where $w(n)$ captures the combined complexities of $\odot$ and $\curlyvee$, and c and k respectively determines the number of subproblems created and recursively solved.

In our synthesis context, we assume $w(n)$ to have simple polynomial complexity, that is $O(n^m)$ for some $m \geq 0$, since it is difficult to relate logarithmic complexities to syntax. We use a triple (m, k, c) to denote the *complexity budget* for a divide-and-conquer algorithm, from which (through the Master Theorem) its asymptotic complexity can be calculated. In a typical divide-and-conquer algorithm, there is usually a tension between the complexity of divide and join functions. If the algorithm does more work upfront, to perform a favourable division, then the task of combining will become simpler. Conversely, if it performs a cheap division, a more elaborate join will be required. *Quick sort* and *merge sort* can be respectively considered instances of the two scenarios. We use this insight to enumerate different possible solutions to synthesis. The $O(n^m)$ total cost for divide and join operations is computed as the combination of the complexities $O(n^{m_\curlyvee})$ for division and $O(n^{m_\odot})$ for join.

5.2 Synthesis Paradigm

Figure 5 illustrates a precise instantiation of the schema presented earlier in Figure 2, incorporating a complexity budget and the specifications introduced in Section 4. Our synthesis algorithm maintains an internal budget for the join operation, which is initialized to the smallest possible value $\mathcal{B}_\odot = (m_\odot, k, c) = (0, 1, 2)$. The algorithm attempts to synthesize a solution within budget $\mathcal{B}_\odot$, and if it fails then it

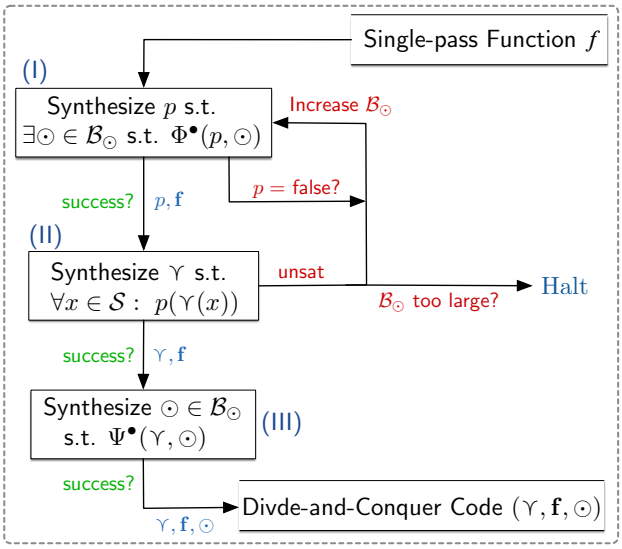


Figure 5. Phased Synthesis Schema

increases the budget until it reaches a limit, where the divide-and-conquer code will end up with higher computational complexity than the input implementation. At the high level, the algorithm proceeds in three phases:

(I) The *weakest divide predicate* p is synthesized that satisfies the specification $\exists\odot : \Phi^\bullet(p,\odot)$. At this stage, the algorithm does not synthesize an implementation for $\odot$ but rather guarantees its existence within budget $\mathcal{B}_\odot$. An algorithm for (I) is the key (algorithmic) contribution of this paper and appears in Section 6.

(II) A divide operation $\curlyvee$ *matching* p is synthesized. $\curlyvee$ is the lowest (time) complexity divide operation that satisfies $\forall x \in \mathcal{S} : p(\curlyvee(x))$. As a direct consequence of the conceptual contributions presented in Section 4, this phase can be efficiently implemented using a straightforward syntax-guided synthesis routine.

(III) An implementation of $\odot$ (within budget $\mathcal{B}_\odot$) is synthesized such that $\Psi^\bullet(\curlyvee,\odot)$ holds. Similar to (II), this step can be implemented using a straightforward syntax-guided synthesis routine.

The loop between steps (I) and (II) may succeed several times, for increasing values of $\mathcal{B}_\odot$, and therefore, it can enumerate many valid solutions when more than one exist.

If step (I) fails to discover a predicate p, the default value *false* is returned which triggers the step to be repeated with a higher budget $\mathcal{B}_\odot$. The fact that p is *the weakest* predicate implicitly guarantees that it satisfies $\chi^\bullet$, and therefore, it does not have to explicitly appear as part of $\Phi^\bullet(p,\odot)$.

The algorithm also produces a *lifting* of f to guarantee the existence of $\odot$ in step (I), if one is required. In other words, the f as it appears in $\Phi^\bullet(p,\odot)$ may be the original f or a lifting $\mathbf{f}$ (of f) that facilitates the existence of the $\odot$. Our proposed algorithm for synthesis of p in Section 6 also accommodates the computation of $\mathbf{f}$, if necessary.

In step (II), the algorithm attempts to synthesize $\curlyvee$ such that total complexity of divide-and-conquer algorithm based on the budget $(max(m_\odot, m_\curlyvee), k, c)$ is at most as computationally expensive as f. A failure in this step means that a divide function matching the predicate p cannot be synthesized. If step (II) fails, then $\mathcal{B}_\odot$ is increased so that a different predicate is produced in step (I).

$\mathcal{B}_\odot = (m_\odot, k, c)$ is increased first by incrementing k until $k = c$, and then by incrementing m until the complexity of f is reached, and finally by incrementing c. Note that there is no theoretical bound on c, but it is often a small constant and therefore a small bound is preset for it in this loop. The loop terminates when $\mathcal{B}_\odot$ reaches its limit.

By the end of step (II), the algorithm has synthesized a *divide* operation $\curlyvee$ that satisfies the specification $\exists\odot : \Psi^\bullet(\curlyvee,\odot)$ such that the combined cost of $\curlyvee$, known since it has been synthesized, and $\odot$, known through $\mathcal{B}_\odot$, does not surpass the asymptotic complexity of f. Therefore, step (III) is guaranteed to succeed.

The solution space of possible divide-and-conquer algorithms in each iteration subsumes that of the previous iteration since $\mathcal{B}_\odot$ is increased. Therefore, a predicate solution from an earlier iteration is a valid solution for a later iteration. However, a new predicate, admitted through a bigger $\mathcal{B}_\odot$, is strictly weaker than a predicate from an earlier iteration . This is precisely why to ensure progress, we actively seek the weakest predicate that satisfies the constraints in step (I). This also guarantees that upon termination, the algorithm explores all possible divide-and-conquer solutions with a join function of polynomial time complexity within acceptable range.

6 Deductive Recursion Synthesis

This section presents an algorithm for step (I) in Figure 5. The goal is to find a divide predicate p such that there is a join function $\odot$ within a given budget $\mathcal{B}_\odot = (m_\odot, k, c)$ that satisfies $\Phi^\bullet(p,\odot)$, that is (for $c = 2$):

$$\forall x, y \in \mathcal{S}^2 : p(x,y) \Rightarrow f(x \bullet y) = f(x) \odot f(y). \quad (7)$$

Key insight. In the above specification, f is the known recursive function, p is an unknown *recursive* predicate and $\odot$ is an unknown recursive function. The idea is to use the recursive definition of f to infer the recursive definition of p (and $\odot$). We do this by induction on the two f input parameters x and y, and the two $\odot$ input parameters $f(x)$ and $f(y)$ (unless they are scalars). Starting with empty lists, one can solve for p and $\odot$ for lists of increasing sizes, and then extrapolate a recursive definition for p (and for $\odot$). Since f is rightwards single-pass, it is sufficient to perform induction only on y and on $f(x)$, but not on x. We start with an intuitive explanation of the algorithm through an example, and then present the formal details.

Recall example POP from Section 2. We illustrate how a recursive definition of p may be discovered for the 2-way

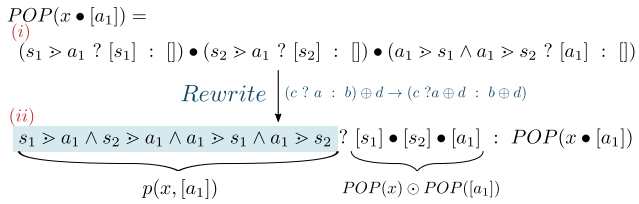


Figure 6. Expression of the unfolding POP($x \bullet [a_1]$)

divide in one of the divide-and-conquer instances discussed for POP. To simplify the presentation of this instance, we assume that we know $\odot = \bullet$ a priori, even though this is not generally the case. We start by doing induction on y and POP(x). We let $y = [a_1]$ and $y = [a_1, a_2]$ for two different instances, and POP(x) = $[s_1, s_2]$. Note that the list elements are symbolically denoted by variables a_1, a_2, s_1, and s_2.

Expression *(i)* in Figure 6 is the result of inlining the recursive definition of POP($x \bullet [a_1]$). Since x is fixed, one can view POP($x \bullet [a_1]$) as the *unfolding* of function POP starting from POP(x). Observe that expression *(i)* in no way resembles the format of Equation 7, and the expression of p cannot be guessed from it. Using a few simple rewrite rules, specifically $(c\ ?\ a : b) \oplus d \rightarrow c\ ?\ a \oplus d : b \oplus d$ and the factoring of conditionals, expression *(i)* can be rewritten to expression *(ii)* [1]. Note that in expression *(ii)*, the then-expression is equal to POP(x) • POP($[a_1]$). Therefore, if the highlighted expression is true, then POP($x \bullet [a_1]$) = POP(x)•POP($[a_1]$), which makes the highlighted expression our best conjecture for $p(x, [a_1])$. Yet, one *unfolding* is not sufficient for deducing a recursive definition for p. Therefore, we repeat this process with y as a list of two elements, which will produce a conjecture for $p(x \bullet [a_1, a_2])$. If we compute the expression for POP($x \bullet [a_1, a_2]$) and rewrite it, we get:

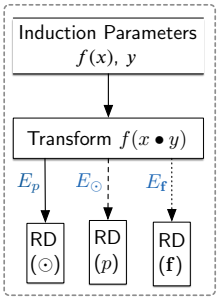
$$\begin{aligned}\text{POP}(x \bullet [a_1, a_2]) = &\big(\neg(a_1 \geqslant a_2) \vee p(x, [a_1])\big) \wedge \big(\neg(a_2 \geqslant a_1) \vee \\ &(s_1 \geqslant a_2 \wedge s_2 \geqslant a_2 \wedge a_2 \geqslant s_1 \wedge a_2 \geqslant s_2)\big)\ ? \\ &\text{POP}(x) \bullet \text{POP}([a_1, a_2]) : \text{POP}(x \bullet [a_1, a_2])\end{aligned}$$

The expression for $p(x, [a_1, a_2])$ is again identifiable from the conditional operator at the root. Observe that both expressions are instances of the more general pattern $\forall r \in \text{POP}(y), \forall s \in \text{POP}(x), s \geqslant r \wedge r \geqslant s$, which is precisely the recursive definition that our algorithm extrapolates from these two instances through a *recursion discovery* (RD) step. We say well-formed expressions like these are in *normal form*. $\Phi^\bullet$ can be transformed to:

$$\forall (x, y) \in \mathcal{S}^2 : f(x \bullet y) = p(x, y)\ ?\ f(x) \odot f(y) : f(x \bullet y) \quad (8)$$

where p appears as a conditional subexpression of the right-hand side, rather than as a precondition in Equation 7. This facilitates the identification of normal forms through the transformation of the expression of $f(x \bullet y)$.

We made the simplifying assumption that $\odot = \bullet$, but in general, it is unknown. Therefore, instead of knowing that POP(x) • POP($[a_1, a_2]$) is the join expression, we have to characterize the shape of valid join expressions. Then, based on Equation 8, a guess is made for a subexpression representing p based on the expression under the condition having the right shape.

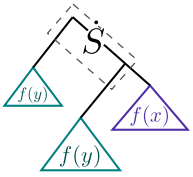


The diagram on the right summarizes the procedure. First, based on induction parameters of increasing size, the expressions of the unfoldings of $f(x \bullet y)$ are transformed to normal forms (see Section 6.1). The relevant sets of subexpressions for p, $\odot$, and a possible lifting $\mathbf{f}$ are extracted from the normal forms for successive unfoldings (see Section 6.2). Synthesizing p is the main goal of this procedure. In some instances, when the transformation works well, the set $E_\odot$ is precise enough so that $\odot$ can be discovered through recursion discovery. But, this is not always the case, and the only guarantee of this step is its existence within budget $\mathcal{B}_\odot$. Lifting is done when required, and the transformation produces the candidate set $E_\mathbf{f}$ (see Section 6.3). Finally, a recursion discovery (RD) subroutine extrapolates recursive definitions for p, $\odot$, and $\mathbf{f}$ respectively out of the sets of expressions E_p, $E_\odot$, and $E_\mathbf{f}$. (see Section 6.4).

6.1 Normal Forms

We first present a characterization of the expression (a normal form) of $\odot$ informed by a budget $\mathcal{B}_\odot$.

$\mathcal{B}_\odot$-normal form. A $\mathcal{B}_\odot$-normal form intuitively describes the shape of the expression of $f(x) \odot f(y)$. To characterize the unfolded expression of a join within budget $\mathcal{B}_\odot$ we define $\mathcal{B}_\odot$-normal forms parameterized by the budget, and the input expressions of the join $f(x)$ and $f(y)$. For a budget $\mathcal{B}_\odot = (m_\odot, k, c)$, the normal form illustrated on the right characterizes the expression of the join in the form of the expression skeleton $\dot{S}$. The leaves completing the skeleton are the inputs to $\odot$ which should be filled with $f(x)$ or $f(y)$. If $\dot{S}$ is meant to characterize a join within budget $\mathcal{B}_\odot$, then it can admit at most k inputs parameters. For example, since both $f(x)$ and $f(y)$ appear in the expression on the right, it is only in normal form for $k = 2$.



The join should be computable in $O(n^{m_\odot})$ time. Recall that normal forms are defined for expressions of fixed size resulting from unfoldings on inputs of fixed size, as the example in the beginning of this section suggests. We define a notion of *cost* for these expressions such that when a general recursive $\odot$ is synthesized using the normal form, we will have the guarantee that $\odot \in O(n^{m_\odot})$. An expression is in normal form if it adheres to a particular shape and has a particular *cost*.

An *expression skeleton* of degree k, denoted $\dot{S}^k$, is an abstract syntax tree (AST) described by the grammar on the

[1] Appendix C.1 spells out the rewriting steps for the interested reader.

right, where the leaves are constants or indexed holes $??_i$ with $1 \leq i \leq k$. Given a set of input expressions $E = \{e_i\}_{1 \leq i \leq k}$, $\dot{S}^k[E]$ denotes the expression constructed by replacing the hole $??_i$ in $\dot{S}^k$ by expression e_i, for all $1 \leq i \leq k$. In our algorithm, $\dot{S}^k$ characterizes the *shape* of a join of arity k, and the e_i's stand for the inputs to the join function, for instance $f(x)$ and $f(y)$. Note that skeletons have a fixed degree k, since k is the number of parameters of $\odot$ fixed by the budget.

$\dot{S}^k = \dot{S}^k \ominus \dot{S}^k \mid \dot{S}^k \lessdot \dot{S}^k$
$\mid \neg \dot{S}^k \mid \dot{S}^k \bullet \dot{S}^k$
$\mid \dot{S}^k.m$
$\mid \dot{S}^k[j]$ where $j \in \mathbb{N}$
$\mid \dot{S}^k \; ? \; \dot{S}^k : \dot{S}^k$
$\mid ??_i$ where $0 < i \leq k$
$\mid$ true $\mid$ false $\mid n \in int$

$\ominus$: arithmetic or boolean operator.
$\lessdot$: comparison operator.
$.m$: field acessor.

The cost associated to each skeleton is a vector $\vec{m}$ of length c (the number of partitions produced by $\curlyvee$). Semantically, it represents the conjecture that $\dot{S}^k[E]$ is computable in $O(n^{max(\vec{m})})$[2] time for arbitrary inputs E of size n. Note that G is a bounded expression, and the e_i that the algorithm considers are bounded, at each step of the inductive reasoning. Yet, the final join function has to have the right time complexity over arbitrary-sized inputs.

Intuitively, one can establish the complexity of the join operator by examining the candidate skeletons over different induction steps. The skeleton from the previous induction step and its associated cost forms a *context* that is used as a parameter to determine the cost of the new skeleton for the current induction step.

The *context* consists of a skeleton $\dot{S}^k_{prev}$, a cost conjecture $\vec{m}$, and an identifier $0 < i_c \leq c$ for the induction parameter expanded in the current induction step. Initially, $\vec{m} = \vec{0}$, and a skeleton of minimal size is assumed in the first induction step. The cost $\vec{m}$ of a skeleton $\dot{S}^k$ is determined based on the subexpression relation between $\dot{S}^k$ and $\dot{S}^k_{prev}$. If $\dot{S}^k$ is not changed in the current induction step, one can infer that its computation takes constant time with respect to the current induction parameter i_c. Otherwise, $\dot{S}^k_{prev}$ must appear as a subexpression of $\dot{S}^k$, since the target of synthesis is a recursive function. If there is a new hole $??_i$ in $\dot{S}^k$, which is not part of $\dot{S}^k_{prev}$, then the induction component $\mu(i)$ that corresponds to the hole is updated (μ maps the input of the skeleton to induction components). Otherwise, the current induction component is updated. Intuitively, since an extra subexpression appears in $\dot{S}^k$, there is an additional computation step required for one induction step, and the computation takes linear time. The algorithm is illustrated on the right, but it is missing a few cases, which seem to be uncommon in practice and did not occur in the synthesis of any of our benchmarks. The complete algorithm, listing all cases, appears in Appendix C.2.

if $\dot{S}^k = \dot{S}^k_{prev}$ **then**
  $\vec{m}[i_c] = 0$;
else if *No new hole in* $\dot{S}^k$ **then**
  $\vec{m}[i_c] = 1$;
else if *New hole* $??_i$ *in* $\dot{S}^k$ **then**
  $\vec{m}[\mu(i)] = 1$;
...;

[2] $max(\vec{v})$ is the maximum of the components of vector $\vec{v}$.

Definition 6.1 ($\mathcal{B}_\odot$-normal form)**.** An expression e is in $\mathcal{B}_\odot$-normal form in context C, for a budget $\mathcal{B}_\odot = (m_\odot, k, c)$, with respect to a family of expressions $E = \{e_i\}_{1<i\leq k}$, iff there exists a skeleton $\dot{S}^k$ such that $e = \dot{S}^k[E]$ and $max(\vec{m}) = m_\odot$, where $\vec{m}$ is the cost of $\dot{S}^k$ in context C.

We say that an expression is $\mathcal{B}_\odot$**-normalizable** in context C with respect to E if it can be rewritten to an expression in $\mathcal{B}_\odot$-normal form in context C with respect to E. The context is only mentioned explicitly if it is relevant.

Multi-way conditional expression. If $p(x, y) \equiv true$, i.e. any division is acceptable, then $f(x \bullet y)$ is $\mathcal{B}_\odot$-normalizable with respect to $\{f(x), f(y)\}$. But, if a special division is necessary, then the shape of the expression $p(x, y)?f(x) \odot f(y) : f(x \bullet y)$ (from Equation 8) hints at the fact that only a subtree of the AST, after rewriting, is in $\mathcal{B}_\odot$-normal form. This is the subexpression that appears under the *then* branch of the conditional expression.

Definition 6.2. An expression $e = \{e_i \textbf{ if } b_i \mid i \in I\}$ is a multi-way conditional expression (MC-expression) with branch conditions $\{b_i\}_{i\in I}$, if branch expressions $\{e_i\}_{i\in I}$ do not contain any

***Example* 6.3.** Let $\uparrow$ denote the infix operator returning the maximum of two values. We use a computation of the longest increasing subsequence (LIS) of a list of integers as our running example in this section. The single-pass function $\mathsf{LIS} : [int] \to int \times int \times int$ with signature $(cl, ml, prev)$, where ml is the length of the longest increasing subsequence and cl is the length of the longest increasing suffix, is defined as (for any sequence x, state s, and integer a):

$$\mathsf{LIS}([]) = (0, 0, -\infty) \qquad \mathsf{LIS}(x \bullet [a]) = \mathsf{LIS}(x) \oplus a$$
$$s \oplus a = \textbf{let } cl = s.prev < a \; ? \; s.cl + 1 : 0 \textbf{ in } (cl, cl \uparrow s.ml, a)$$

We consider the second unfolding starting from $\mathsf{LIS}(x) = (cl_0, ml_0, prev_0)$, with input $[a_1, a_2]$. The expression of $\mathsf{LIS}(x \bullet [a_1, a_2]).ml$ is translated to a MC-expression with 4 branches:

$$\begin{array}{lll}
1: & ml_0 \uparrow cl_0 + 1 \uparrow cl_0 + 1 + 1 & \textbf{if } (a_1 < a_2) \wedge (prev_0 < a_1) \\
2: & ml_0 \uparrow cl_0 + 1 \uparrow 0 & \textbf{if } (a_1 \geq a_2) \wedge (prev_0 < a_1) \\
3: & ml_0 \uparrow 0 \uparrow 0 + 1 & \textbf{if } (a_1 < a_2) \wedge (prev_0 \geq a_1) \\
4: & ml_0 \uparrow 0 \uparrow 0 & \textbf{if } (a_1 \geq a_2) \wedge (prev_0 \geq a_1)
\end{array}$$

The expression of the divide predicate p intuitively corresponds to the subset of branches where the expressions under guards match the expressions of the function $\odot$. Formally, in a MC-expression $e = \{e_i \textbf{ if } b_i \mid i \in I\}$, a boolean expression b **isolates** the subset of branches $I' \subseteq I$ iff $\forall i \in I' : b_i \implies b$ and $\forall i \in I \setminus I' : b_i \implies \neg b$.

6.2 Expression Transformation

Given an unfolding of $f(x \bullet y)$, we first transform it into an MC-expression. Then we check which branches have expressions that are $\mathcal{B}_\odot$-normalizable by attempting to normalize them. The expression of p is exactly the expression isolating the subset of branches that are successfully rewritable to $\mathcal{B}_\odot$-normal forms, and the expression of $\odot$ is the skeleton defining the normal forms.

***Example* 6.4.** Recall Example 6.3 and suppose that we have a constant time budget for $\odot$. In the second induction step, the two inputs of the join are $\{(cl_0, ml_0, prev_0), \mathsf{LIS}([a1, a2])\}$. Branch 3 and 4 expressions can be rewritten to a $\mathcal{B}_\odot$-normal form, witnessed by the skeleton $\dot{S}^2 = ??_1.ml \uparrow ??_2.ml$, which can be computed in constant time with cost $(0, 0)$ (accounting for the context from previous step).

However, there is no normal form of cost $(0, 0)$ for the branches 1 and 2: the branches contain subexpressions of the form $cl_0 + 1 + \ldots$ that grow in size with each induction step, so the inferred cost in these branches is $(0, 1)$.

Branches (3,4) are normalizable for a constant-time budget, and the expression of the predicate is identified by isolating those branches: $p(\mathsf{LIS}(x), [a_1, a_2]) = a_1 \leq prev_0$. ⌟

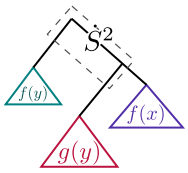
Any expression in the program can be transformed to an MC-expression. The first step consists in using a strongly normalizing rewrite system, with rules similar to the ones used in the introductory example of this section (complete list in Appendix A.3). The expression generated by the rewrite system is an MC-expression. Then a solver eliminates the branches that are infeasible; that is, each branch with index i such that $\neg b_i$ is valid is removed.

The $\mathcal{B}_\odot$-normal form generalizes the constant normal forms and recursive normal forms defined in [9]. Normalizing a symbolic expression to a $\mathcal{B}_\odot$-normal form can be done by small adjustments in the rewriting process from [9], which in turn is a relatively standard cost-based rewrite system. We list the rewrite rules used in Appendix A.2. Note that the context for $\mathcal{B}_\odot$-normalization depends on the branch of the MC-expression. Each branch has a matching branch at the previous induction step, from which the context is taken.

With an ideal normalization process, the predicate expressions synthesized are guaranteed to correspond to the expressions of the weakest predicate that ensures a join operator $\odot$ exists within budget $\mathcal{B}_\odot$, since at each unfolding stage, all the branches that are $\mathcal{B}_\odot$-normalizable are isolated. But, since reachability of an existing $\mathcal{B}_\odot$-normal form is undecidable [8, 9], the weakness of p cannot be theoretically guaranteed for all inputs. This also implies that the join cannot be always synthesized by our algorithm.

6.3 Automatic Lifting

During the process of identifying $\mathcal{B}_\odot$-normalizable subexpressions, instead of discovering a clean $\mathcal{B}_\odot$-normal form, the expression sometimes normalizes to a tree of the form illustrated on the right. There are leaves corresponding to $f(x)$ and $f(y)$ as before, but there is also a leaf that corresponds to subcomputations not already performed by f. The figure labels this as a new function $g(y)$. The normal form implies that the join operator needs access to the result of $g(y)$ to produce the overall result. Hence, f needs to lifted to compute g in addition to its original computation.



Normalization of a single branch of an MC-expression can have three possible outcomes: (i) success (i.e. no lifting required), (ii) lifting required, or (iii) failure, when the cost of the expression surpasses the budget. One can aim for a solution based on all branches in class (i) (with no lifting), or for one based on all branches in classes (i) and (ii) to produce the weakest predicate supported by the lifting.

***Example* 6.5.** Recall Example 6.4; a predicate p is discovered without a need for lifting (i.e. branches 3 and 4 belong in class (i)). Suppose now that we aim to identify *all* branches as normalizable; this will lead to $p \equiv true$, then $\vee$ being the random splitting, an instance of a MapReduce solution. For this a lifting is required, since subexpressions of the form $cl_0 + (1 + \ldots)$, which appear branches 1 and 2, have to be precomputed to maintain the possibility of a constant-time join. The exact expression depends on the condition $a_1 < a_2$, therefore we derive the auxiliary computation $g_1([a_1, a_2]) = a_1 < a_2 \ ? \ 1 + 1 \ : \ 1$. Additionally, the condition isolating branches (3,4), $a_1 \leq prev_0$ has to be available for join, the extra auxiliary $g_2([a_1, a_2]) = a_1$ is also required. Therefore, branches 1 and 2 belong in class (ii). ⌟

A similar deductive-style automated lifting was introduced for lists in [8] and extended to multidimensional lists in [9]. But with the aid of MC-expressions, we can infer strictly more expressive auxiliary computation in this paper, in particular we can synthesize *conditional auxiliary* computations. More details about the procedure and an example are presented in Appendix C.3.

6.4 Recursion Discovery

The goal of recursion discovery is to deduce the recursive definition of a function from its unfoldings. In [8, 9], a procedure is proposed for solving this exact problem. It operates by using subtree isomorphisms to identify different stages of a recursive computation in an input set of expressions. We apply their procedure as a black-box in two instances: the divide predicate and the lifting discovery.

At each step of the induction process, the unfoldings of the rightwards single-pass function f from an initial state $f(x)$ on sequence y are transformed to expressions of the form $p(f(x), y) \ ? \ f(x) \odot \mathbf{f}(y) \ : \ f(x \bullet y)$. With an ideal normalization process, this would allow to identify the unfoldings

of $\odot$, p and g, a function that computes the information required in addition to f in the lifting $\mathbf{f}$. As noted in Section 6.1 no such ideal procedure exists, and in practice we can only identify the expressions of $p(f(x), y)$ and $g(y)$, but not $\odot$, for the different values of $f(x)$ and y during the induction process.

Recursion discovery is used to produce a recursive definition of $p_0 : D \times \mathcal{S} \to \text{Bool}$ from the unfoldings of p_0, i.e. the expressions of the form $p_0(f(x), y)$, for different inputs $f(x)$ and y. The function p_0 is defined recursively by an operator $\otimes$ such that $p_0(f(x), y \bullet [a]) = p_0(f(x), y) \otimes (f(x), y, a)$. Note that $p_0(f(x), y)$ may be false, which means no corresponding divide can be discovered. Once p_0 is discovered, the divide predicate p is simply defined as $p(x, y) = p_0(f(x), y)$.

Recursion discovery is also used to discover a recursive definition of a lifting of f when necessary. It produces a function g from the expressions $g(y)$ (for different values of y) identified after normalization, which is then tupled with f to form a lifting $\mathbf{f}$.

***Example* 6.6.** In Example 6.5, we identified bounded expressions that correspond to the required lifting. From a set of such expressions, recursion discovery deduces a recursive definition for g_1, with signature $(cond, aux)$:

$$g_1(x \bullet [a]) = \textbf{let } c \ = \ g_1(x).cond \wedge (f(x).prev < a) \textbf{ in}$$
$$\textbf{let } b = g_1(x).aux \textbf{ in } (c, c \ ? \ b + 1 : b)$$

The final lifting of LIS is $\text{LIS}'(x) = (\text{LIS}(x), g_1(x), head(x))$ since *head* is the trivial result of recursion discovery from the unfoldings of g_2 in Example 6.5. ⌟

7 Synthesizing Divide and Join

Once the divide predicate p is successfully synthesized, the two remaining tasks are the synthesis of the join function and the synthesis the divide function using p as its specification. These are simple tractable synthesis problems for search-based synthesis tools, which is precisely the reason why we decomposed the problem in this particular manner. We briefly explain each task (at the high level) for the sake of the completeness of the paper.

7.1 Divide Function Synthesis

We use syntax-guided synthesis [3] to synthesize a divide function according to specification $\forall z \in \mathcal{S}, p(\vee(z)) \wedge \chi(\vee)$ (from Section 4). For a SyGuS solution, the search space for synthesis has to be defined.

If $p(x, y) \equiv \text{true}$, then the inverse of concatenation is a valid solution. Incidentally, it is the only valid constant-time divide function. If $p(x, y) \not\equiv \text{true}$, we assume that $\vee$ has at least linear time complexity. By analyzing the predicate p, we can distinguish whether only a splitting divide (Definition 4.4) is required, or a partition divide needs to be synthesized. If the predicate $p(x, y)$ is a condition on a prefix of its second argument y, then a splitting divide is synthesized. The divide is constructed as a function that scans the input sequence from a random location, until the condition on the prefix starting from the location is met, at which point the sequence is split into the current prefix and the suffix.

Otherwise, the divide is *sketched* [3] as a partition function that operates in two phases, first by selecting one or more *pivots*, and then partitioning the elements of the inputs list according to their relation to these pivots. For a given sketch, a number q of pivots is fixed. For a budget (m, k, c), the unknowns are the q pivot selection functions and $c - 1$ two-way partition functions using the pivots. If no solution for a given q is found, q can be increased. 2 pivots seemed to be sufficient to cover all our benchmarks. The time complexity is at least linear, but can be higher if the selection of pivot requires super-linear time. While none of our benchmarks required a super-linear pivot selection function, we successfully experimented with synthesizing one with our tool to test its robustness. The example is a pivot constrained to be the median of a list. Detailed descriptions of these sketches are given in Appendix C.4.

7.2 Join Operator Synthesis

With $\vee$ known, the specification is simplified to $\forall z \in \mathcal{S}$, $f(z) = f(\vee(z).1) \odot f(\vee(z).2)$. In general, this synthesis problem is *identical* to a similar problem that was solved in [9], with good theoretical guarantees. We do not repeat that contribution here. Whenever the procedure described in Section 6 succeeds in producing the join, the synthesis step in (III) is effectively reduced to a bounded verification of the already discovered $\odot$, effectively checking that the divide-and-conquer algorithm with the divide synthesized in step (II) and the join operator inferred at step (I) is functionally equivalent to the original function. For example, concatenation, as the join for the two-way divide solution of POP, is inferred at the divide predicate synthesis step.

8 Experimental Results

Our approach is implemented as an extension of the tool Parsynt [20], which accepts as input C-like iterative programs with loops or functional programs written in Scheme. It is implemented in OCaml [16] and uses Z3 [6] as SMT solver and Rosette [25] as syntax-guided synthesis solver. All experiments were run on a desktop with two 8-core Intel Xeon E5-2620 and 32GB of RAM running Ubuntu 18.04.

To the best of our knowledge Parsynt is the only fully automatic tool that can synthesize divide-and-conquer programs of the class described in this paper from a reference implementation. A number of tools, including BIGλ [23], and Casper [1], synthesize various types of MapReduce [7] programs. The MapReduce model is too restrictive for splitting or partitioning divides, and all the tools mentioned fail to synthesize a solution for POP example from Section 2 or LIS example from Section 6. GraSSP [11] goes slightly beyond MapReduce and parallelizes single pass array computations,

but the most expressive class they target is subsumed by our solutions with splitting divides. An earlier version of Parsynt [9] targets nested loops and performs lifting, but the divide operations are limited to the inverse of concatenation. We use the most difficult benchmarks from GraSSP as some of our simplest benchmarks.

Benchmarks. The theoretical results of Section 4 suggest a classification of the algorithms targeted in this paper into those with *splitting divides* (e.g. LIS) and those with *partitioning divides* (e.g. POP). We evaluate the efficacy and efficiency of Parsynt in synthesizing divide-and-conquer algorithms for two sets of benchmarks, one for each category. We collected our benchmarks from algorithm textbooks and related work on divide-and-conquer programming. These are non-trivial iterative algorithms for which equivalent divide-and-conquer algorithms according to Equation 1 exist. Those that have a solution with a partitioning divide are listed in Table 1 and those with a splitting divide in Table 7(a). The first set includes sophisticated algorithms, where some have several distinct divide-and-conquer implementations, synthesized using different budgets. The second set is composed of single-pass algorithms computing counts or maximal lengths of subsequences that have a given property.

Performance of Parsynt. Table 1 and Figure 7(a) report the synthesis times for each phase of the synthesis procedure. For each synthesis task in Table 1, we report the synthesis times separately for each budget that led to a synthesized solution. When the predicate synthesized is trivially true, there is no need to synthesize a divide; these cases are denoted by †. The synthesis times range from a few seconds to up to 26 min. The solutions with three-way divides ($c = 3$) require significantly more time to be synthesized. This is due to the fact that the size of the bounded model required by the synthesis needs to be increased to take in account the increase in dimension. All input implementations have $O(n^2)$ time complexity[3], and therefore the synthesized solutions with $O(n \log n)$ complexity are highly non-trivial, on par with the three solutions of POP discussed in Section 2.

In Table 7(a), we report the times spent in the predicate and the join synthesis steps. Note that the predicate synthesis times listed are the combined times for both lifting and the predicate synthesis step (which take place in one step).

The benchmarks for which GraSSP [11] can also synthesize a solution are highlighted in blue. Note that Parsynt synthesizes two solutions for each benchmark against one for the other tool. The synthesis times are overall small for these benchmarks, but Parsynt still illustrates a time advantage (see Appendix D.1). The rest of our benchmarks are rejected by other tools (including GraSSP), because they require *lifting* in the form of addition of one or more conditional accumulators with a non-trivial accumulation operation.

[3]For the k-largest benchmark we consider the case where k is large.

Table 1. Partitioning Divides: Columns p, $\curlyvee$, $\odot$ present the synthesis time (in seconds) for the respective functions, and column $O(?)$ lists the time complexity of the synthesized code. † indicates that no divide needs to be synthesized, a greyed cell signals that lifting was required, and × means that Parsynt fails.

	$\mathcal{B}_\odot$	p	$\curlyvee$	$\odot$	$O(?)$
Sorting	(0,2,2)	4.5	1.1	5.1	$n \log n$
k-largest	(0,2,2)	3.4	1.2	120	$n \log n$
Closest pair	(0,2,2)	5.6	1.2	10	$n \log n$
Intersecting intervals	(0,2,2)	12	54	30	$n \log n$
	(0,3,3)	31	421	1.5	$n \log n$
Histogram	(0,2,2)	4.1	1.1	25.3	$n \log n$
	(2,2,2)	3.0	†	9.4	n^2
POP	(0,2,2)	5.3	69	8.7	$n \log n$
	(1,2,2)	6.2	20	240	$n \log n$
	(2,2,2)	3.1	†	12	n^2
	(0,2,3)	35	1560	91	$n \log n$
Minimal points	(0,2,2)	5.0	64	10.5	$n \log n$
	(1,2,2)	6.4	21.5	206	$n \log n$
	(2,2,2)	3.0	†	11.5	n^2
	(0,2,3)	35	1430	87.0	$n \log n$
Quadrant orthogonal convex hull	(0,2,2)	5.2	67	13	$n \log n$
	(1,2,2)	6.7	24.5	201	$n \log n$
	(2,2,2)	3.0	†	12	n^2
	(0,2,3)	35	1540	88	$n \log n$
Orthogonal convex hull	(1,2,2)	×	×	×	$n \log n$
	(2,2,2)	6.1	†	24	n^2
Encircling set	(0,2,2)	×	×	×	$n \log n$

Quality of the synthesized code. The synthesized implementations for the benchmarks in Table 1 belong to one of the two categories: (1) the synthesized divide-and-conquer algorithm has a strictly *lower asymptotic complexity* than the input sequential code (any row with $O(n \log n)$ complexity) or (2) its asymptotic complexity is the same (about 22% of the cases). In Section 2, we discussed how different input distributions may result in a preference for one solution over another, for the latter case.

The solutions with a splitting divide lead to scalable parallel implementations, as showcased in Figures 7(b) and 7(c). In Figure 7(b) we compare the speedup of the different parallel implementations of the benchmarks of Table 7(a) with varying number of threads, for an input of 10^{10} integers with indivisible blocks of 100 elements in average. For each benchmark, we have two solutions: one with a splitting divide (plotted with a continuous line) and one with a lifting (dashed line). Both implementations scale in parallel with comparable performance gains. The relative speedups for these can also depend on the input data composition. Figure 7(c) compares the relative speedups of the two implementations of LIS, for varying sizes of increasing sequences in the input. When increasing sequences are long, the splitting divide implementation performs significantly worse than the one with lifting. This observation generalizes across all benchmarks that have splitting divide and lifting solutions, and makes a case for why synthesizing two solutions is useful.

(a)

	$p + \curlyvee$	$\odot$
Count 1(0+)	0.1	1.2
Count 1(0*)2	0.1	1.4
Count 1*2*3	0.2	3.1
Max dist. between 1s	0.2	2.6
Max sum between 0s	0.2	1.8
LIS	0.1	1.3
Largest peak	0.1	1.2
Longest 1*	0.1	1.2
Longest 1(0*)2	0.2	3.1
Longest even 0*	0.2	19.1
Longest odd (01)*	0.2	25.0
Longest 1(0*)2(0*)3	×	×

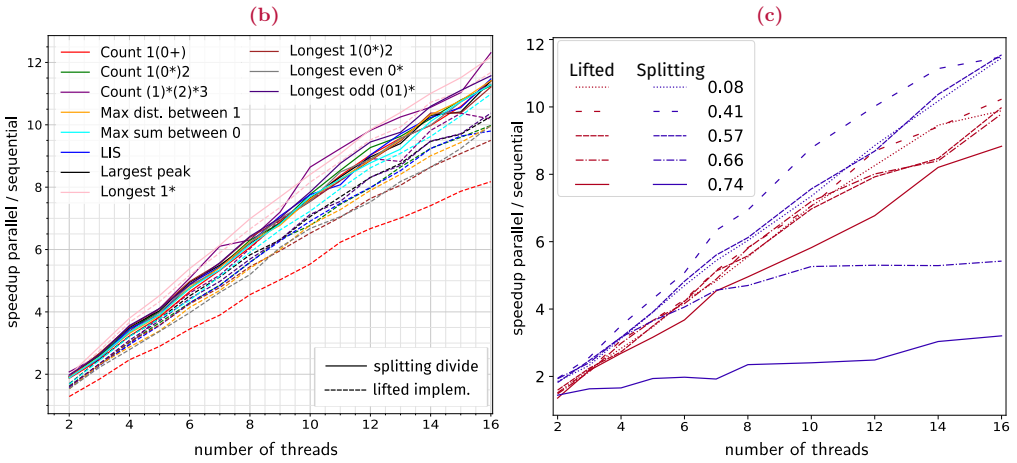


Figure 7. Splitting Divides: Table (a) lists the synthesis times (in seconds) of the benchmarks with splitting divides. × indicates that the tool failed to find a solution. Figure (b) illustrates the speedups of the parallel implementations. Figure (c) compares the speedups of two parallel implementations of the LIS benchmark, for different ratios of size of increasing sequences to total size of input.

Limitations. Each table also lists the benchmarks that underline the limitations of the various steps of our synthesis process. We indicate by × the synthesis steps for which Parsynt fails. For example, in Table 1 the solution for $\mathcal{B}_{\odot} = (1, 2, 2)$ of the *orthogonal convex hull* benchmark, which requires a complex lifting, could not be automatically synthesized. The tool cannot synthesize the divide for the *encircling set* benchmark because it involves the synthesis of a function with non-linear arithmetic operations. The benchmark *Longest 1(0*)2(0*)3* in the last row of Table 7(a) (where the code computes the length of the longest substring matching the regular expression $1(0^*)2(0^*)3$) admits a splitting divide, but the deductive synthesis procedure cannot infer a divide predicate due to the large number of new variables required to compute the predicate. The reader can refer to Appendix E.1 where we outline how difficult it is to derive these divide-and-conquer algorithms, even manually.

9 Related Work

There is a vast body of work on program synthesis. Here we only survey the work related to divide-and-conquer synthesis. Map-reduce is one of the most popular subclasses of divide-and-conquer, which formally relies on the computation being a list homomorphism, the precise class of functions that can be written as a composition of a *map* and a *reduction* (cf. first homomorphism theorem). The literature on divide-and-conquer synthesis can be divided into two categories based on the class of input computations targeted: (1) those with list homomorphisms as input, with the aim of synthesizing efficient map-reduce [7] programs [1, 14, 21, 23], (2) those that go beyond list homomorphisms [8, 9, 11, 15, 19, 22], and target code with more dependencies. In category (2), the techniques in [8, 9, 11] *synthesize list homomorphisms* through some variation of *lifting*, the approach in [22] uses symbolic execution at runtime and to identify and defer dependencies, and Bellmania [15] targets input programs in the style of dynamic programming and orchestrates an efficient execution schedule to accommodate the dependencies. A direct comparison with work in [1, 9, 11, 23] with respect to the class of input programs appears in Section 8.

Derivation of list homomorphisms includes approaches based on the third homomorphism theorem [13, 14, 19], function composition [12], and quantifier elimination [18], as well as those based on recurrence equations [4]. These techniques are either not fully automatic, or rely on additional guidance from the programmer beyond the input sequential code. In contrast, the techniques in [8, 9, 11, 22] derive list homomorphisms automatically through *lifting*. The lifting performed in [9] is strictly the most general one and subsumes the rest.

The class of divide-and-conquer algorithms targeted in this paper is strictly more general than list homomorphisms, and therefore more general than both categories (1) and (2) of work mentioned earlier. To the best of our knowledge, no prior work targets a class as general as this automatically. In [24], *manual synthesis* of general classes is discussed.

10 Conclusion

We solve a program synthesis problem with three unknown components, related through a single specification, by decomposing it into tractable subtasks. The key takeaways are: (1) our deductive synthesis technique based on *induction*, *rewriting*, and *recursion discovery* is a powerful method for the synthesis of recursive code where another recursive code is available as the functional specification, and (2) an imperfect deductive synthesis algorithm can be utilized as

an oracle producing powerful hints, which can be used to decompose a monolithic synthesis problem with multiple unknowns into a sequence of more tractable synthesis problems over subsets of these unknowns.

Our deductive synthesis module differs from the classic one in that instead of operating on the source code, it manipulates the results of its symbolic evaluation. Small variations in code may result in the same symbolically evaluated term, and therefore, the technique is more robust with respect to syntactic variations in the input implementations.

Our approach in decomposing the monolithic divide-and-conquer specification is in the spirit of multi-abduction [2]: a specification with multiple unknowns is decomposed into specifications for each unknown. The problem is that in this domain, like many others, individual maximal specifications for each component do not exist; a stronger specification on divide would imply less work to be done at join time. We exploit the structure of the problem to effectively enumerate all admissible pairs of specifications, by relying on the complexity of the join function to guide this enumeration.

References

[1] Maaz Bin Safeer Ahmad and Alvin Cheung. 2018. Automatically Leveraging MapReduce Frameworks for Data-Intensive Applications. In *Proceedings of the 2018 International Conference on Management of Data*. ACM, New York, NY, USA, 1205–1220. https://doi.org/10.1145/3183713.3196891

[2] Aws Albarghouthi, Isil Dillig, and Arie Gurfinkel. 2016. Maximal Specification Synthesis. In *Proceedings of the 43rd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*. ACM, New York, NY, USA, 789–801. https://doi.org/10.1145/2914770.2837628

[3] Rajeev Alur, Rastislav Bodík, Eric Dallal, Dana Fisman, Pranav Garg, Garvit Juniwal, Hadas Kress-Gazit, P. Madhusudan, Milo M. K. Martin, Mukund Raghothaman, Shambwaditya Saha, Sanjit A. Seshia, Rishabh Singh, Armando Solar-Lezama, Emina Torlak, and Abhishek Udupa. 2015. Syntax-Guided Synthesis. In *Dependable Software Systems Engineering*. 1–25. https://doi.org/10.1109/FMCAD.2013.6679385

[4] Yosi Ben-Asher and Gadi Haber. 2001. Parallel Solutions of Simple Indexed Recurrence Equations. *IEEE Trans. Parallel Distrib. Syst.* 12, 1 (Jan. 2001), 22–37. https://doi.org/10.1109/71.899937

[5] Jon Louis Bentley, Dorothea Haken, and James B Saxe. 1978. *A General Method for Solving Divide-and-Conquer Recurrences*. Technical Report. https://doi.org/10.1145/1008861.1008865

[6] Leonardo De Moura and Nikolaj Bjørner. 2008. Z3: An efficient SMT solver. In *International conference on Tools and Algorithms for the Construction and Analysis of Systems*. Springer, 337–340. https://doi.org/10.1007/978-3-540-78800-3_24

[7] Jeffrey Dean and Sanjay Ghemawat. 2008. MapReduce: Simplified Data Processing on Large Clusters. *Commun. ACM* 51, 1 (Jan. 2008), 107–113. https://doi.org/10.1145/1327452.1327492

[8] Azadeh Farzan and Victor Nicolet. 2017. Synthesis of Divide and Conquer Parallelism for Loops. In *Proceedings of the 38th ACM SIGPLAN Conference on Programming Language Design and Implementation*. ACM, 540–555. https://doi.org/10.1145/3140587.3062355

[9] Azadeh Farzan and Victor Nicolet. 2019. Modular Divide-and-conquer Parallelization of Nested Loops. In *Proceedings of the 40th ACM SIGPLAN Conference on Programming Language Design and Implementation*. ACM, 610–624. https://doi.org/10.1145/3314221.3314612

[10] Azadeh Farzan and Victor Nicolet. 2021. From Iterative Implementations to Single-pass Functions. (2021). http://www.cs.toronto.edu/~azadeh/resources/papers/functional_translation.pdf (manuscript).

[11] Grigory Fedyukovich, Maaz Bin Safeer Ahmad, and Rastislav Bodik. 2017. Gradual Synthesis for Static Parallelization of Single-Pass Array-Processing Programs. In *Proceedings of the 38th ACM SIGPLAN Conference on Programming Language Design and Implementation*. Association for Computing Machinery, New York, NY, USA, 572–585. https://doi.org/10.1145/3140587.3062382

[12] Allan L. Fisher and Anwar M. Ghuloum. 1994. Parallelizing Complex Scans and Reductions. In *Proceedings of the ACM SIGPLAN 1994 Conference on Programming Language Design and Implementation*. 135–146. https://doi.org/10.1145/773473.178255

[13] Alfons Geser and Sergei Gorlatch. 1997. Parallelizing Functional Programs by Generalization. In *Proceedings of the 6th International Joint Conference on Algebraic and Logic Programming*. 46–60. https://doi.org/10.1017/S0956796899003536

[14] Sergei Gorlatch. 1996. Systematic Extraction and Implementation of Divide-and-Conquer Parallelism. In *Proceedings of the 8th International Symposium on Programming Languages: Implementations, Logics, and Programs*. 274–288. https://doi.org/10.1007/3-540-61756-6_91

[15] Shachar Itzhaky, Rohit Singh, Armando Solar-Lezama, Kuat Yessenov, Yongquan Lu, Charles Leiserson, and Rezaul Chowdhury. 2016. Deriving divide-and-conquer dynamic programming algorithms using solver-aided transformations. In *ACM SIGPLAN Notices*, Vol. 51. ACM, 145–164. https://doi.org/10.1145/3022671.2983993

[16] Xavier Leroy, Damien Doligez, Alain Frisch, Jacques Garrigue, Didier Rémy, and Jérôme Vouillon. 2018. The OCaml system release 4.07: Documentation and user's manual. (2018).

[17] Zohar Manna and Richard Waldinger. 1979. Synthesis: dreams → programs. *IEEE Transactions on Software Engineering* 4 (1979), 294–328. https://doi.org/10.1109/TSE.1979.234198

[18] Akimasa Morihata and Kiminori Matsuzaki. 2010. Automatic Parallelization of Recursive Functions Using Quantifier Elimination. In *Functional and Logic Programming, 10th International Symposium, FLOPS 2010, Sendai, Japan, April 19-21, 2010. Proceedings*. 321–336. https://doi.org/10.1007/978-3-642-12251-4_23

[19] Kazutaka Morita, Akimasa Morihata, Kiminori Matsuzaki, Zhenjiang Hu, and Masato Takeichi. 2007. Automatic Inversion Generates Divide-and-conquer Parallel Programs. In *Proceedings of the 28th ACM SIGPLAN Conference on Programming Language Design and Implementation*. 146–155. https://doi.org/10.1145/1273442.1250752

[20] Victor Nicolet. 2017. Parsynt. https://github.com/victornicolet/parsynt

[21] Cosmin Radoi, Stephen J. Fink, Rodric Rabbah, and Manu Sridharan. 2014. Translating Imperative Code to MapReduce. In *Proceedings of the 2014 ACM International Conference on Object Oriented Programming Systems Languages & Applications*. 909–927. https://doi.org/10.1145/2660193.2660228

[22] Veselin Raychev, Madanlal Musuvathi, and Todd Mytkowicz. 2015. Parallelizing User-defined Aggregations Using Symbolic Execution. In *Proceedings of the 25th Symposium on Operating Systems Principles*. 153–167. https://doi.org/10.1145/2815400.2815418

[23] Calvin Smith and Aws Albarghouthi. 2016. MapReduce Program Synthesis. In *Proceedings of the 37th ACM SIGPLAN Conference on Programming Language Design and Implementation*, Vol. 51. 326–340. https://doi.org/10.1145/2980983.2908102

[24] Douglas R Smith. 1985. Top-down synthesis of divide-and-conquer algorithms. *Artificial Intelligence* 27, 1 (1985), 43–96. https://doi.org/10.1016/0004-3702(85)90083-9

[25] Emina Torlak and Rastislav Bodík. 2013. Growing solver-aided languages with rosette. In *ACM Symposium on New Ideas in Programming and Reflections on Software, Onward! 2013, part of SPLASH '13, Indianapolis, IN, USA, October 26-31, 2013*. 135–152. https://doi.org/10.1145/2509578.2509586